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SHRINKAGE ESTIMATOR BETWEEN BAYES AND MAXIMUM LIKELIHOOD ESTIMATORS

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ABSTRACT

In this paper the $\widetilde{\theta_1}$ shrinkage estimator between MLE estimator and Bayes estimator. Also $\widetilde{\theta_2}$, the shrinkage estimator between Bayes estimator and MLE estimator, and $\widetilde{\theta_3}$ shrinkage estimator between $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$ for estimating the parameter of exponential distribution of life time is presented. Through simulation study the performance of this estimator was compared to the standard Bayes and MLE estimator with respect to the mean square error (MSE) . We found the $\widetilde{\theta_3}$ shrinkage estimator between $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$ is the best estimator.

KEYWORDS: Exponential Distribution, Bayes Method, Maximum Likelihood Estimation and Shrinkage Estimator

INTRODUCTION

One of the most useful and widely exploited models is the exponential distribution. Epstein,(1984) remarks that the exponential distribution plays as important role in life experiments as that played by the normal distribution in agricultural experiments. Maximum likelihood estimation has been the widely used method to estimate the parameter of an exponential distribution. Lately Bayes method has begun to get the attention of researchers in the estimation procedure. The only statistical theory that combines modeling inherent uncertainty and statistical uncertainty is Bayesian statistics. The theorem of Bayes provides a solution on how to learn from data. Related to survival function and by using Bayes estimator, Elli and Rao,(1986), estimated the shape and scale parameters of the Weibull distribution by assuming a weighted squared error loss function. They minimized the corresponding expected loss with respect to a given posterior distribution.**[1]**

Consider the one parameter exponential life time distribution $f(t;\theta)=\frac{1}{\theta}\exp(-\frac{t}{\theta})$.We find Jeffery prior by taking $g(\theta)\propto\sqrt{I(\theta)}$, where $I(\theta)=-nE\left(\frac{\partial^2 \ln f(t,\theta)}{\partial\theta^2}\right)=\frac{n}{\theta^2}$. Taking $g(\theta)\propto\frac{\sqrt{n}}{\theta}$, $g(\theta)=k\frac{\sqrt{n}}{\theta}$, with k a constant.

The joint probability density function $f(t_{1,}t_2,...,t_n,\theta)$ is given by $H(t_1,...,t_n,\theta)=\prod_{i=1}^{n} f(t_i,\theta)g(\theta)=L(t_1,...,t_n|\theta)g(\theta)$, where

$$L(t_1,...,t_n|\theta)=\prod_{i=1}^{n} f(t_i|\theta)=\frac{1}{\theta^n}\exp\left(-\frac{\sum_{i=1}^{n}t_i}{\theta}\right)\quad H(t_1,...,t_n,\theta)=\frac{1}{\theta^n}\exp\left(-\frac{\sum_{i=1}^{n}t_i}{\theta}\right)\frac{k\sqrt{n}}{\theta}=\frac{k\sqrt{n}}{\theta^{n+1}}\exp\left(-\frac{\sum_{i=1}^{n}t_t}{\theta}\right).$$

The marginal probability density function of θ given the data $(t_1,t_2,...,t_n)$ is

$$p(t_1,...,t_n)=\int H(t_1,...,t_n,\theta)d\theta = \int_0^\infty \frac{k\sqrt{n}}{\theta^{n+1}}\exp\left(-\frac{\sum_{i=1}^n t_i}{\theta}\right)d\theta=\frac{\left(k\sqrt{n}\right)(n-1)!}{\left(\sum_{i=1}^n t_i\right)^n}$$, where $$\int_0^\infty \frac{1}{\theta^{n+1}}e^{\frac{-\sum t_i}{\theta}}\,d\theta=\frac{(n-1)!}{(\sum t_i)^n}$$

The conditional probability density function of θ given the data $(t_1,t_2,...,t_n)$ is given by

$$\Pi(\theta|t_1,...,t_n)=\frac{H(t_1,...,t_n,\theta)}{p(t_1,...,t_n)}=\frac{\frac{k\sqrt{n}}{\theta^{n+1}}\exp\left(-\frac{\sum_{i=1}^n t_i}{\theta}\right)}{\frac{k\sqrt{n}}{\left(\sum_{i=1}^n t_i\right)^n}(n-1)!}=\frac{\exp\left(-\frac{\sum_{i=1}^n t_i}{\theta}\right)}{\theta^{n+1}}\frac{\left(\sum_{i=1}^n t_i\right)^n}{(n-1)!}$$

By using squared error loss function $\ell\left(\theta-\theta\right)=c\left(\theta-\theta\right)^2$, we can obtain the Risk function, such that

$$R\left(\theta-\theta\right)=\int_0^\infty \ell\left(\theta,\theta\right)\Pi\left(\theta|t_1,...,t_n\right)d\theta=c\theta^2-2c\theta\int_0^\infty \frac{\left(\sum_{i=1}^n t_i\right)}{(n-1)!}\theta^{-n}\exp\left(-\frac{\sum_{i=1}^n t_i}{\theta}\right)d\theta+\zeta(\theta),$$

where $\zeta(\theta)=\frac{c\,(\sum t_i)^2}{(n-1)\,(n-2)}$. Let $\frac{\partial R(\theta,\theta)}{\partial\theta}=0$, then the Bayes estimator is $\theta_{B_1}=\frac{\left(\sum_{i=1}^n t_i\right)^n}{(n-1)!}\int_0^\infty \theta^{-n}\exp\left(-\frac{\sum_{i=1}^n t_i}{\theta}\right)d\theta=\frac{\sum_{i=1}^n t_i}{n-1}$

[1].

Where $E\left(\widetilde{\theta_B}\right)=\frac{n}{n-1}\theta$ and $var\left(\widetilde{\theta_B}\right)=\frac{n}{(n-1)^2}\theta$,then $MSE\left(\widetilde{\theta_B}\right)=\frac{n+1}{(n-1)^2}\theta^2$

In this paper, we proposed $\widetilde{\theta_1}$ shrinkage estimator between MLE estimator and Bayes estimator and calculate $\widetilde{\theta_2}$, the shrinkage estimator between Bayes estimator and MLE estimator, and then $\widetilde{\theta_3}$ shrinkage estimator between $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$.

Shrinkage Estimator between MLE Estimator and Bayes Estimator

let $\widetilde{\theta_1}=P_1\widetilde{\theta_m}+(1-P_1)\widetilde{\theta_B}$

$$E\left(\widetilde{\theta_1}-\theta\right)^2=P_1{}^2E\left(\widetilde{\theta_m}-\theta\right)^2+(1-P_1)^2E\left(\widetilde{\theta_B}-\theta\right)^2$$
$$+P_1(1-P_1)E\left(\widetilde{\theta_m}-\theta\right)\left(\widetilde{\theta_B}-\theta\right)$$

Then

$$MSE\left(\widetilde{\theta_1}\right)=P_1{}^2MSE\left(\widetilde{\theta_m}\right)+(1-P_1)^2MSE\left(\widetilde{\theta_B}\right)+2P_1(1-P)$$
$$\left[E\left(\widetilde{\theta_m}\widetilde{\theta_B}\right)-E\left(\widetilde{\theta_m}\theta\right)-E\left(\widetilde{\theta_B}\theta\right)+E\theta^2\right]$$

Let $\frac{\partial MSE(\widetilde{\theta_1})}{\partial P_1} = 0$, then this implies that the value of P_1 which minimizes $MSE(\widetilde{\theta_1})$ is

$$P_1 = \frac{MSE(\widetilde{\theta_B}) - E(\widetilde{\theta_m}\widetilde{\theta_B}) + E(\widetilde{\theta_m}\theta) + E(\widetilde{\theta_B}\theta) - E\theta^2}{MSE(\widetilde{\theta_m}) + MSE(\widetilde{\theta_B}) - 2E(\widetilde{\theta_m}\widetilde{\theta_B}) + 2E(\widetilde{\theta_m}\theta) + 2E(\widetilde{\theta_B}\theta) - 2E\theta^2}$$

We have $MSE(\widetilde{\theta_m}) = \frac{\theta^2}{n}, MSE(\widetilde{\theta_B}) = \frac{n+1}{(n-1)^2}\theta^2$

$$E(\widetilde{\theta_m}) = E\frac{\sum t_i}{n} = \theta,\ E(\widetilde{\theta_B}) = E\frac{\sum t_i}{n-1} = \frac{n}{n-1}\theta$$

Then

$$P_1 = \frac{\frac{n+1}{(n-1)^2}\theta^2 - E\left(\frac{\sum t_i}{n}\frac{\sum t_i}{n-1}\right) + E\left(\frac{\sum t_i}{n}\theta\right) + E\left(\frac{\sum t_i}{n-1}\theta\right) - E\theta^2}{\frac{\theta^2}{n} + \frac{n+1}{(n-1)^2}\theta^2 - 2E\frac{(\sum t_i)^2}{n(n-1)} + 2E\left(\frac{\sum t_i}{n}\theta\right) + 2E\left(\frac{\sum t_i}{n-1}\theta\right) - 2E\theta^2}$$

$$P_1 = \frac{n^2 + n}{2n^2 - n + 1}$$

Then the shrinkage estimator between MLE estimator and Bayes estimator is

$$\widetilde{\theta_1} = \left(\frac{n^2 + n}{2n^2 - n + 1}\right)\widetilde{\theta_m} + \left(\frac{n^2 - 2n + 1}{2n^2 - n + 1}\right)\widetilde{\theta_B}$$

Shrinkage Estimator between Bayes Estimator and MLE Estimator

Let $\widetilde{\theta_2} = P_2\widetilde{\theta_B} + (1 - P_2)\widetilde{\theta_m}$

Now we calculate $E(\widetilde{\theta_2} - \theta)^2$, $MSE(\widetilde{\theta_1})$ and $\frac{\partial MSE(\widetilde{\theta_2})}{\partial P_2} = 0$, then this implies that the value of P_2 which minimizes $MSE(\widetilde{\theta_2})$ is

$$P_2 = \frac{MSE(\widetilde{\theta_m}) - E(\widetilde{\theta_B}\widetilde{\theta_m}) + E(\widetilde{\theta_B}\theta) + E(\widetilde{\theta_m}\theta) - E\theta^2}{MSE(\widetilde{\theta_B}) + MSE(\widetilde{\theta_m}) - 2E(\widetilde{\theta_B}\widetilde{\theta_m}) + 2E(\widetilde{\theta_B}\theta) + 2E(\widetilde{\theta_m}\theta) - 2E\theta^2}$$

Also , we have $MSE(\widetilde{\theta_m}) = \frac{\theta^2}{n}, MSE(\widetilde{\theta_B}) = \frac{n+1}{(n-1)^2}\theta^2$

$$E(\widetilde{\theta_m}) = E\frac{\sum t_i}{n} = \theta,\ E(\widetilde{\theta_B}) = E\frac{\sum t_i}{n-1} = \frac{n}{n-1}\theta$$

Then

$$P_2 = \frac{\frac{\theta^2}{n} - E\left(\frac{\sum t_i}{n-1}\frac{\sum t_i}{n}\right) + E\left(\frac{\sum t_i}{n-1}\theta\right) + E\left(\frac{\sum t_i}{n}\theta\right) - E\theta^2}{\frac{\theta^2(n+1)}{(n-1)^2} + \frac{\theta^2}{n} - 2E\frac{(\sum t_i)^2}{n(n-1)} + 2E\left(\frac{\sum t_i}{n}\theta\right) + 2E\left(\frac{\sum t_i}{n-1}\theta\right) - 2E\theta^2}$$

$$P_2 = \frac{n^2 - 2n + 1}{2n^2 - n + 1}$$

Then the shrinkage estimator between Bayes estimator and MLE estimator is

$$\widetilde{\theta_2} = \left(\frac{n^2 - 2n + 1}{2n^2 - n + 1}\right)\widetilde{\theta_B} + \left(\frac{n^2 + n}{2n^2 - n + 1}\right)\widetilde{\theta_m}$$

Shrinkage Estimator between $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$

$$let\ \widetilde{\theta_3} = P_3\widetilde{\theta_1} + (1 - P_3)\widetilde{\theta_2}$$

$$P_3 = \frac{MSE\widetilde{\theta_2} - E\left(\widetilde{\theta_1}\widetilde{\theta_2}\right) + E\left(\widetilde{\theta_1}\theta\right) + E\left(\widetilde{\theta_2}\theta\right) - E\theta^2}{MSE\left(\widetilde{\theta_1}\right) + MSE\left(\widetilde{\theta_2}\right) - 2E\left(\widetilde{\theta_1}\widetilde{\theta_2}\right) + 2E\left(\widetilde{\theta_1}\theta\right) + 2E\left(\widetilde{\theta_2}\theta\right) - 2E(\theta^2)}$$

To get P_3 , we can find $MSE\left(\widetilde{\theta_2}\right), MSE\left(\widetilde{\theta_1}\right)$, $E\left(\widetilde{\theta_1}\right)$ and $E\left(\widetilde{\theta_2}\right)$ such that

$$MSE\left(\widetilde{\theta_2}\right) = \frac{n^2 - 2n + 1}{2n^2 - n + 1}MSE\left(\widetilde{\theta_B}\right) + \frac{n^2 + n}{2n^2 - n + 1}MSE\left(\widetilde{\theta_m}\right)$$

Then $MSE\left(\widetilde{\theta_2}\right) = \frac{4n^6 - 8n^5 + 18n^4 - 17n^3 + 12n^2 - 5n + 2}{4n^6 + n^3 + 3n^2 - n + 1}\theta^2$

Also , we find $MSE\left(\widetilde{\theta_1}\right)$, that is

$$MSE\left(\widetilde{\theta_1}\right) = \frac{n^2 + n}{2n^2 - n + 1}MSE\left(\widetilde{\theta_m}\right) + \frac{n^2 - 2n + 1}{2n^2 - n + 1}MSE\left(\widetilde{\theta_B}\right)$$

Then $MSE\left(\widetilde{\theta_1}\right) = \frac{n^6 - 2n^4 + 2}{4n^6 - 12n^5 + 17n^4 - 16n^3 + 10n^2 - 4n + 1}\theta^2$

$$E\left(\widetilde{\theta_1}\right) = \frac{n^2 + n}{2n^2 - n + 1}E\left(\widetilde{\theta_m}\right) + \frac{n^2 - 2n + 1}{2n^2 - n + 1}E\left(\widetilde{\theta_B}\right)$$

$$= \frac{n^5 - n^4 - n^3 + n^2}{4n^5 - 6n^4 + 4n^3 - 4n^2 + 3n - 1}\theta$$

And $E\left(\widetilde{\theta_2}\right) = \frac{4n^5 - 6n^4 + 2n^3 - 4n^2 - n}{4n^5 - 8n^4 + 5n^3 - 5n^2 + n - 1}\theta$

Now ,we calculate $P_{3,}$ such that

$$P_3 = \left[\left(\frac{A_1}{B_1}\theta^2 + \frac{C_1}{D_1}\theta^2 - \theta^2\right)\left(\frac{A_2}{B_2}\theta^2 - \frac{C_2}{D_2}\theta^2 + \frac{F_2}{D_2}\theta^2 - 2\theta^2\right)\right],$$

$$= \left[\left(\frac{A_1}{B_1} + \frac{C_1}{D_1} - 1\right)\left(\frac{A_2}{B_2} - \frac{C_2}{D_2} + \frac{F_2}{D_2} - 2\right)\right]$$

where

$$A_1 = 16n^{16} + 64n^{15} + 28n^{14} - 256n^{13} + 246n^{12} - 82n^{11} - 100n^{10}$$

$$+215n^9 - 222n^8 + 142n^7 - 65n^6 + 57n^5 - 18n^4 + 10n^3 - 5n^2 + 2n$$

$$B_1 = 64n^{16} - 288n^{15} + 336n^{14} - 136n^{13} + 368n^{12} - 444n^{11} + 394n^{10}$$

$$-200n^9 + 162n^8 - 223n^7 + 197n^6 - 150n^5 + 79n^4 - 21n^3 + 15n^2 + 2n + 1$$

$$C_1 = 200n^{10} - 60n^9 - 3n^8 - 66n^7 + 57n^6 - 32n^5 + 8n^4 - 8n^3 + n$$

$$D_1 = 16n^{10} - 56n^9 + 84n^8 - 98n^7 + 34n^6 - 30n^5 + 13n^4 - 20n^3 + 4n^2 - 4 + 1$$

$$A_2 = 20n^{12} - 72n^{11} + 203n^{10}$$

$$-449n^9 + 701n^8 - 820n^7 + 739n^6 - 530n^5 + \quad 271n^4 - 76n^3 + 4n^2 + n$$

$$B_2 = 16n^{12} - 48n^{11} + 68n^{10}$$

$$-60n^9 + 40n^8 - 39n^7 + 55n^6 - 66n^5 + \quad 27n^4 - 17n^3 + 9n^2 - 5n + 1$$

$$C_2 = 8n^{10} - 20n^9 + 8n^8 + 4n^7 - 4n^6 + 12n^5 - 2n^4 - n^3$$

$$D_2 = 16n^{10} - 56n^9 - 12n^8 - 88n^7 + 38n^6 - 78n^5 + 41n^4 - 20n^3 + 4n^2 - 4n + 1$$

$$F_1 = 16n^{10} - 48n^9 + 60n^8 - 68n^7 + 14n^6 - 38n^5 + 24n^4 - 10n^3 + n^2 + n$$

Simulation Study

In this simulation study, we have chosen n=30,60,90 to represent small, moderate and large sample size, several values of parameter θ =0.5, 1.1, 1.5 .The number of replication used was R=1000.

The simulation program was written by using Matlab program. After the parameter was estimated , mean square error (MSE) is calculated to compare the methods of estimation, where MSE $(\theta) = \frac{\sum_{i=1}^{1000}(\theta - \theta)^2}{R}$

The results of the simulation study are summarized and tabulated in Table 1 for the MSE of the five estimators for all sample sizes and θ values respectively.

The order of the estimator is from the best (smallest MSE) to the worst (largest MSE). It is obvious from these tables, shrinkage estimator between $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$, $\widetilde{\theta_3}$, is the best estimator in most of the cases.

Table 1: The Ordering of the Estimators with Respect to MSE

n	θ	$\widetilde{\theta}_B$	$\widetilde{\theta}_m$	$\widetilde{\theta}_1$	$\widetilde{\theta}_2$	$\widetilde{\theta}_3$
30	0.5	0.0114	0.0102	0.0104	0.0105	0.0102
	1.1	0.0113	0.0101	0.0114	0.0119	0.0100
	1.5	0.0111	0.0110	0.0115	0.0116	0.0109
60	0.5	0.0121	0.0110	0.0112	0.0115	0.0108
	1.1	0.0119	0.0109	0.0111	0.0112	0.0107
	1.5	0.0119	0.0110	0.0110	0.0109	0.0107
90	0.5	0.0117	0.0118	0.0108	0.0108	0.0107
	1.1	0.0116	0.0162	0.0108	0.0107	0.0105
	1.5	0.0115	0.0155	0.0107	0.0107	0.0105

CONCLUSIONS

The new estimator $\widetilde{\theta_3}$ that is shrinkage estimator between $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$, is the best estimator when compared to standard Bayes, MLE estimator, and other estimators. We can easily conclude that MSE shrinkage estimator between $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$ decrease with an increased of sample size.

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